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ON THE STABILITY OF THE EULER–LAGRANGE FUNCTIONAL EQUATION

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ABSTRACT. S. M. Ulam imposed before the Mathematics Club of the University of Wisconsin (in 1940) the following problem: "Give conditions in order for a linear mapping near an approximately linear mapping to exist". Then D. H. Hyers (1941) solved this problem. In this paper we state and prove a theorem for an analogous problem for approximately non-linear Euler–Lagrange mappings.

Theorem *Let X be a normed linear space, Y be a Banach space, and $f : X \to Y$. If there exist $a, b : 0 \le a + b < 2$, and $c_2 \ge 0$ such that*

$$(1) \qquad \|f(x+y) + f(x-y) - 2 \cdot [f(x) + f(y)]\| \le c_2 \cdot \|x\|^a \cdot \|y\|^b$$

for all $x, y \in X$, then there exists a unique non-linear mapping $N : X \to Y$ such that

$$(2) \qquad \|f(x) - N(x)\| \le c \cdot \|x\|^{a+b}$$

and

$$(2)' \qquad N(x+y) + N(x-y) = 2 \cdot [N(x) + N(y)]$$

for all $x, y \in X$, where $c = c_2/(4 - 2^{a+b})$.

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Note that a mapping $N : X \to Y$ satisfying $(2)'$ is called *Euler–Lagrange mapping*, and a mapping $f : X \to Y$ satisfying (1) is *approximately Euler–Lagrange mapping*.

Proof (of existence). Inequality (1) and $y = x$ imply

$$\|4^{-1} \cdot f(2x) - f(x)\| \leq 4^{-1} \cdot c_2 \cdot \|x\|^{a+b}. \tag{3}$$

Similarly from (3) we get

$$\|4^{-2} \cdot f(2^2x) - 4^{-1} \cdot f(2x)\| \leq 4^{-1} \cdot c_2 \cdot 2^{a+b-2} \cdot \|x\|^{a+b}. \tag{3$'$}$$

Applying triangle inequality and adding (3) and $(3)'$ we find

$$\begin{aligned} \|4^{-2} \cdot f(2^2x) - f(x)\| &\leq \|4^{-2} \cdot f(2^2x) - 4^{-1} \cdot f(2x)\| \\ &\quad + \|4^{-1} \cdot f(2x) - f(x)\| \\ &\leq 4^{-1} \cdot c_2 \cdot \sum_{i=0}^{1} 2^{i(a+b-2)} \cdot \|x\|^{a+b}. \end{aligned} \tag{3$''$}$$

More generally, the following lemma holds.

Lemma 1. *In space X,*

$$\|4^{-n} \cdot f(2^n x) - f(x)\| \leq 4^{-1} \cdot c_2 \cdot \sum_{i=0}^{n-1} 2^{i(a+b-2)} \cdot \|x\|^{a+b}, \tag{4}$$

for some $c_2 \geq 0$ and for any positive integer n.

To prove Lemma 1 we work by induction on n. For $n = 1$, the result is obvious from (3). We assume then that (4) holds for $n = k$ and prove that (4) is true for $n = k + 1$. Indeed from (4) and $n = k$ and $2x = z$ we find

$$\|4^{-k} \cdot f(2^k z) - f(z)\| \leq 4^{-1} \cdot c_2 \cdot \sum_{i=0}^{k-1} 2^{i(a+b-2)} \cdot \|z\|^{a+b}, \quad \text{or}$$

$$\|4^{-(k+1)} \cdot f(2^{k+1}x) - 4^{-1} \cdot f(2x)\| \leq 4^{-1} \cdot c_2 \cdot \sum_{i=0}^{k-1} 2^{(i+1)(a+b-2)} \cdot \|x\|^{a+b}$$

or

$$\|4^{-(k+1)} \cdot f(2^{k+1}x) - 4^{-1} \cdot f(2x)\| \leq 4^{-1} \cdot c_2 \cdot \sum_{i=1}^{k} 2^{i(a+b-2)} \cdot \|x\|^{a+b}. \tag{5}$$

Therefore from (3), (4), and triangle inequality we get

$$\|4^{-(k+1)} \cdot f(2^{k+1}x) - f(x)\|$$
$$\leq \|4^{-(k+1)} \cdot f(2^{k+1}x) - 4^{-1} \cdot f(2x)\| + \|4^{-1} \cdot f(2x) - f(x)\|$$
$$\leq 4^{-1} \cdot c_2 \cdot \sum_{i=0}^{k} 2^{i(a+b-2)} \cdot \|x\|^{a+b}, \qquad \text{or}$$

(4) holds for $n = k + 1$, or

$$\|4^{-(k+1)} \cdot f(2^{k+1}x) - f(x)\| \leq 4^{-1} \cdot c_2 \cdot \sum_{i=0}^{k} 2^{i(a+b-2)} \cdot \|x\|^{a+b}. \tag{6}$$

It is clear that (3) and (6) yield (4), completing the proof of Lemma 1.

But

$$\sum_{i=0}^{n-1} 2^{i(a+b-2)} < \sum_{i=0}^{\infty} 2^{i(a+b-2)} = 1/(1 - 2^{a+b-2}) = c_0, \tag{7}$$

because $a, b : 0 \leq a + b < 2$.

Set

$$c = 4^{-1} \cdot c_2 \cdot c_0 = c_2/(4 - 2^{a+b}). \tag{7$'$}$$

Then Lemma 1, (7), and (7)$'$ imply

$$\|4^{-n} \cdot f(2^n x) - f(x)\| \leq c \cdot \|x\|^{a+b} \tag{8}$$

for any $x \in X$, any positive integer n, and some $c_2 \geq 0$.

Lemma 2. *The sequence* $\{4^{-n} \cdot f(2^n x)\}$ *converges.*

To prove that the sequence $\{4^{-n} \cdot f(2^n x)\}$ is a Cauchy sequence we first use (8) and the completeness of Y. In fact, if $i > j > 0$, then

$$\|4^{-i} \cdot f(2^i x) - 4^{-j} \cdot f(2^j x)\| = 4^{-j} \cdot \|4^{-(i-j)} \cdot f(2^{i-j} \cdot h) - f(h)\|, \tag{9}$$

where $h = 2^j x$. From (8) and (9) we get

$$\|4^{-i} \cdot f(2^i x) - 4^{-j} \cdot f(2^j x)\| \leq 4^{-j} \cdot c \cdot \|h\|^{a+b}, \quad \text{or}$$
$$\|4^{-i} \cdot f(2^i x) - 4^{-j} \cdot f(2^j x)\| \leq 4^{-j} \cdot c \cdot \|2^j x\|^{a+b}$$
$$= c \cdot 2^{j(a+b-2)} \cdot \|x\|^{a+b}$$

or

$$\lim_{j\to\infty} \|4^{-i} \cdot f(2^i x) - 4^{-j} \cdot f(2^j x)\| = 0 \tag{10}$$

because $a, b : 0 \le a + b < 2$.

It is obvious now from (10) and the completeness of Y that the sequence $\{4^{-n} \cdot f(2^n x)\}$ converges and therefore the proof of Lemma 2 is complete.

Set

$$N(x) = \lim_{n\to\infty} (4^{-n} \cdot f(2^n x)). \tag{11}$$

It is clear from (1) and (11) that

$$\|f(2^n x + 2^n y) + f(2^n x - 2^n y) - 2[f(2^n x) + f(2^n y)]\| \le c_2 \cdot \|2^n x\|^a \cdot \|2^n y\|^b,$$

or

$$\|4^{-n} \cdot f(2^n(x+y)) + 4^{-n} \cdot f(2^n(x-y)) - 2[4^{-n} \cdot f(2^n x) + 4^{-n} \cdot f(2^n y)]\|$$
$$\le c_2 \cdot 4^{-n} \cdot \|2^n x\|^a \cdot \|2^n y\|^b = c_2 \cdot 2^{n(a+b-2)} \cdot \|x\|^a \cdot \|y\|^b$$

or by taking limits $(n \to \infty)$

$$\|N(x+y) + N(x-y) - 2[N(x) + N(y)]\| = 0$$

for any $x, y \in X$, because $a, b : 0 \le a + b < 2$, or

$$N(x+y) + N(x-y) = 2[N(x) + N(y)]$$

for any $x, y \in X$. However, if we take limits on (8) we obtain (2), completing *the proof of existence of a non-linear Euler-Lagrange mapping* $N : X \to Y$ *satisfying* (2).

Uniqueness. *Let* $M : X \to Y$ *be a non-linear Euler-Lagrange mapping, such that*

$$\|f(x) - M(x)\| \le c' \cdot \|x\|^{a'+b'}, \qquad c' \ge 0, \tag{12}$$

for any $x \in X$, *where* $a', b' : 0 \le a' + b' < 2$ *and* c' *is a constant. If there exists a non-linear Euler-Lagrange mapping* $N : X \to Y$, *then*

$$N(x) = M(x) \tag{13}$$

for any $x \in X$.

To prove (13) we must prove the following

Lemma 3. *If* (2)–(2)′, (12) *and*

$$M(x+y) + M(x-y) = 2[M(x) + M(y)] \tag{12}$$

hold, then

$$\|N(x) - M(x)\| \leq c \cdot m^{a+b-2} \cdot \|x\|^{a+b} + c' \cdot m^{a'+b'-2} \cdot \|x\|^{a'+b'} \tag{14}$$

for all m, *for any* $x \in X$, $a, b : 0 \leq a+b < 2$, *and* $a', b' : 0 \leq a' + b' < 2$.

The required result (14) follows immediately if we use inequalities (2) and (12), the triangle inequality and the fact that

$$N(x) = m^{-2} \cdot N(mx), \qquad M(x) = m^{-2} \cdot M(mx). \tag{15}$$

In fact,

$$\|N(mx) - M(mx)\| \leq \|N(mx) - f(mx)\| + \|f(mx) - M(mx)\|.$$

Then if we apply (2), (12) and (15) we obtain inequality (14) completing the proof Lemma 3.

It is clear now that (14) implies

$$\lim_{m \to \infty} \|N(x) - M(x)\| = 0$$

for any $x \in X$, and thus the proof of (13) is complete. Therefore *the uniqueness part of our theorem* is complete, as well.

Query. *What is the situation in the above theorem in case* $a + b = 2$?

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